A Behavioural Money-Pump Argument for Completeness

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ABSTRACT. According to Completeness, at least one of two prospects is at least as preferred as the other. I present a new money-pump argument that Completeness is a requirement of rationality. In comparison with earlier money-pump arguments for Completeness, the new argument relies on a form of unidimensional stochastic dominance and the behavioural assumption that agents pick in a probabilistic manner when no option is optimal. Moreover, unlike some of the previous arguments, the new argument is based on a forcing money pump, that is, an exploitation scheme where the agent is rationally required at each step to go along with the scheme.

According to Completeness, at least one of two prospects is at least as preferred as the other. Letting '$X \succsim Y$' denote that X is at least as preferred as Y, we can state Completeness as follows:[1]

Completeness $X \succsim Y$, or $Y \succsim X$.

In this paper, I will present a new money-pump argument that Completeness is a requirement of rationality. In comparison with earlier money-pump arguments for Completeness, this new argument relies on a form of unidimensional stochastic dominance and the behavioural assumption that agents pick in a probabilistic manner when no option is optimal.[2] Moreover, unlike some of the previous arguments, the new argument is

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[1] Arrow 1951, p. 13. von Neumann and Morgenstern (1944, pp. 26–7) use a slightly weaker variant of Completeness which allows that a prospect is not at least as preferred as itself.

[2] Compared to the deontic money-pump argument for Completeness in Gustafsson 2022, pp. 24–39 (which is also based on a forcing money pump), the new argument relies on neither the Principle of Rational Decomposition nor the Principle of Future-Choice Independence. See Gustafsson 2022, pp. 28, 31.

based on a *forcing* money pump — that is, an exploitation scheme where the agent is rationally required at each step to go along with the scheme.[3]

Suppose that you violate Completeness by having a preferential gap between prospects A and B. Letting '$X \parallel Y$' denote a preferential gap between X and Y, your preference can be stated as follows:

(1) $A \parallel B$.

A crucial difference between a preferential gap and indifference is that a preferential gap is robust to some mild sourings or sweetenings.[4] A *souring* of a prospect X is a prospect that is just like X except that it is inferior in one dimension that the agent cares about (for instance, the souring may be worse in terms of money). Whereas a *sweetening* of a prospect X is a prospect that is just like X except that it is superior in one dimension that the agent cares about.

If the preferential gap between A and B is only robust to sweetenings or sourings of A (and not B), we relabel A as 'B' and B as 'A'. If the preferential gap between A and B is only robust to sweetenings of B (and not sourings), we relabel one of these sweetenings of B as 'B' and B as 'B^{--}'.[5] Moreover, we assume that a souring of a prospect is less preferred than the original prospect. Hence, letting '$X \succ Y$' denote that X is preferred to Y, we can infer that there will be a souring B^{--} of B such that

(2) $A \parallel B^{--}$, and $B \succ B^{--}$.

[3] Chang's (1997, p. 11) and Broome's (1999, pp. 156–7; 2000, pp. 33–4) money pumps are non-forcing. For the forcing/non-forcing distinction, see Gustafsson and Espinoza 2010, pp. 761–2 and Gustafsson 2022, p. 27. Bradley and Steele (2016, pp. 8–9) claim (without spelling out the details) that agents with incomplete preferences may pay to avoid free information, which is closely related to a money pump. They (2016, p. 25) maintain, however, that such agents (given a suitable choice rule) won't face a forcing money pump. Finally, like the present paper, Bader (2019) discusses agents with probabilistic behaviour in choices between options related by a preferential gap. His cases involve iterated choices between the sourings of the options with the unsoured options only being reachable after having reject a large number of sourings (which each might be selected with some probability). But these cases, as Bader points out, only become arbitrarily close to proper (sure loss) money pumps.

[4] This is analogous to Raz's (1985–1986, p. 120; 1986, pp. 325–6) similar 'mark of incommensurability' for value incomparability.

[5] Accordingly, we don't need the assumption that the robustness to sourings is symmetrical between the relata of the preferential gap, which is needed for the money-pump argument for Completeness in Gustafsson 2022, p. 26.

(We label the souring 'B^{--}' rather than 'B^{-}' to leave notational room for a milder souring of B later on.)

How do you choose between options when no option is preferred to the others? A standard answer is that you pick rather than choose.[6] If one picks in a fully blind way, it seems that the probability that each alternative will be selected is uniform. Picking in that fully blind way, however, makes most sense when the agent's preferential attitudes for the options are exactly symmetrical.[7] Having a preferential gap between two options doesn't necessarily require that one's preferential attitudes are symmetrical between the options. For instance, it seems plausible, given (1), that the agent's preferential comparison of A and B has a slightly different balance than their preferential comparison of A and B^{--}. Plausibly then, the agent is more likely to select A over B^{--} than to select A over B. So we allow (but do not require) that these probability distributions might differ for different pairs of options when the agent probabilistically picks between them. (We will assume, however, that these probability distributions are knowable rather than Knightian.[8] But we'll cover the Knightian case later on.)

Suppose then that, in a choice between A and B, you probabilistically pick A with probability p (where $0 < p < 1$). And, in a choice between A and B^{--}, you probabilistically pick B^{--} with probability q (where $0 < q < 1$). And we assume that neither p nor q is equal to 0 or 1, because, if you certainly choose X in a choice between X and Y, then you effectively prefer X to Y.

Moreover, we assume the following requirement of rationality:[9]

> *The Preferential Behaviour Restriction* The agent's behaviour at a choice between X and Y only depends on the agent's preferential attitudes for X and Y.

[6] Ullmann-Margalit and Morgenbesser 1977, p. 757.

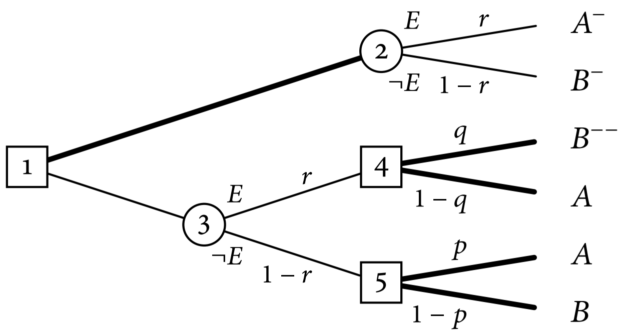
[7] Such symmetrical cases are the kinds of cases for which Ullmann-Margalit and Morgenbesser (1977, pp. 757–9) proposed picking.

[8] See Knight 1921, p. 20.

[9] A notable difference between the money-pump argument for Completeness presented in this paper and that in Gustafsson 2022, pp. 24–39 is that the latter does not need this assumption, relying instead on the, deontic rather than behavioural, principle *Decision-Tree Separability* — that the rational status of the options at a choice node does not depend on other parts of the decision tree than those that can be reached from that node. See Gustafsson 2022, p. 9.

The idea is that your behaviour in a choice between two options should be based on your preferential attitudes for those options. Note that, if the agent has a preferential gap between two options, the relevant preferential attitudes can be something richer than merely having a preferential gap. But the Preferential Behaviour Restriction rules out that the agent's choice behaviour in a choice between two options depends on any parts of a decision problem that can no longer be reached from the current choice node.[10]

Holding fixed your preferential attitudes that we will now try to exploit, consider the following decision problem:



Here, the boxes represent choice nodes and the circles represent chance nodes. At the chance nodes, chance goes up if and only if chance event E happens — both chance nodes depending on the same event. This event E occurs with a probability r such that

$$(3) \qquad r = \frac{p}{p+q}.$$

The potential outcomes if you go up at node 1 (that is, A^- and B^-) will be introduced shortly.

At node 1, you predict that you would probabilistically pick B^{--} at node 4 with probability q and probabilistically pick A at node 5 with probability p. So — letting '$[X, p; Y, q; \dots]$' denote the prospect of X with probability p, Y with probability q, and so on — you predict that, if you were to go down at node 1, you would face prospect D:

$$(4) \qquad D = [B^{--}, rq; A, r(1-q) + (1-r)p; B, (1-r)(1-p)].$$

[10] Hence this requirement rules out resolute choice; see McClennen 1990, pp. 12–13. For a critical discussion of resolute choice, see Gustafsson 2022, pp. 66–74.

Now, consider the following requirement of rationality:[11]

> *The Strong Principle of Unidimensional Stochastic Dominance* If (i) X^- is a souring of X, (ii) $X \succ X^-$, and (iii) $p > 0$, then $[X, p; Y, 1-p] \succ [X^-, p; Y, 1-p]$.

In other words, if a first prospect gives the same probability of each outcome as a second prospect except that in the second prospect some of those outcomes are soured, then you prefer the first prospect to the second. As far as rational requirements concerning uncertainty go, it's a compelling requirement.[12]

By the Strong Principle of Unidimensional Stochastic Dominance, it follows that you prefer the prospect of $[A, r; B, 1-r]$ to D. To see this, note that the probability of ending up with A given D is equal to

$$r(1-q) + (1-r)p = \frac{p}{p+q}(1-q) + (1 - \frac{p}{p+q})p = \frac{p}{p+q} = r.$$

So, given D, the probability of ending up with A is r and the probability of ending up with one of B and B^{--} is $1-r$. And, since D gives you some chance of B^{--}, it follows that $[A, r; B, 1-r]$ dominates D. So we have

(5) $[A, r; B, 1-r] \succ D$.

Next, consider the following requirement of rationality:[13]

> *Unidimensional Continuity of Preference* If $X \succ Y$, then there is a prospect X^- such that (i) X^- is a souring of X and (ii) $X \succ X^- \succ Y$.

If you prefer X to Y, then, it seems, you must prefer X to Y with some margin. So there should some small extent to which we can sour X such

[11] Gustafsson 2022, p. 59. While this principle is compelling, it's a drawback of the present approach that we need to make any assumption specifically about uncertain prospects to defend Completeness, which isn't essentially about uncertainty (although it Completeness, of course, also applies to uncertain prospects). The approach in Gustafsson 2022, pp. 24–9 does not rely on assumptions specifically about uncertain prospects.

[12] This requirement should be plausible even if one is risk-averse. For instance, Buchak (2013, pp. 37–8), who favours risk aversion of the expected-utility-violating variety, accepts a still stronger version.

[13] Gustafsson 2022, p. 5.

that this souring is also preferred to Y (the souring, of course, will be less preferred than X).

From (5), it follows, by Unidimensional Continuity of Preference, that there are sourings of A^- and B^- of A and B respectively such that

(6) $[A, r; B, 1-r] \succ [A^-, r; B^-, 1-r] \succ D.$

So now we have the missing prospects of the decision problem: If were to you go up at node 1, you would end up with A^- if E occured and with B^- if E did not occur.

At node 1, you rely on backward induction — that is, you take into account your predictions about your behaviour at future choice nodes when you make your choice. Since $[A^-, r; B^-, 1-r]$ is the prospect of going up at node 1 and you predict that you would face the less preferred prospect D if you were to go down, you go up at node 1. But then, regardless of whether E occurs, you will end up with a souring of what you would have ended up with if you had gone down at each choice node.

	E occurs	E does not occur
Up at node 1	A^-	B^-
Down at each choice node	A	B

Accordingly, this case is a money pump. We can think of the choice nodes as trading opportunities. At each choice node, you receive a trade offer, which you accept by going up and you turn down by going down. And we can think of you as initially possessing the prospect $[A, r; B, 1-r]$ — that is, what you would get if you were to turn down all trade offers. At node 1, you are offered a trade from $[A, r; B, 1-r]$ to $[A^-, r; B^-, 1-r]$. If you were to turn down that trade and E occurred, you would be offered a trade from A to B^{--} at node 4. And, if you were to turn down the trade at node 1 and E did not occur, you would be offered a trade from B to A at node 5. Since you would go up at node 1, you are guaranteed to pay (letting the unidimensional sourings be monetary) for what you could have walked away with for free.

Note that this money pump can't be blocked by foresight. You know the whole exploitation set-up in advance. And we assumed that you make your choice at node 1 with the help of backward induction. So, even though you have foresight and use backward induction, you prefer going up at node 1. And so you get exploited.[14]

[14] In fact, this decision problem is *BI-terminating* — that is, backward induction only

It may be objected that the plan to walk away from all trade offers isn't available in the relevant sense at node 1, since there is some positive probability that you would accept one of the offers at nodes 4 and 5. But note that what makes the plan to walk away from all offers unavailable in this sense is your incomplete preferential attitudes — which is the target of the argument.[15] So your incomplete preferential attitudes are still to blame for your being exploitable.

It may next be objected that it's unrealistic that an exploiter would have the detailed knowledge of your preferential attitudes that is needed for the exploiter to set-up this money pump. But all money-pump schemes require that the exploiter (to some extent) knows the agent's preferences. What matters for the money-pump argument is not that it is easy for the exploiter to set-up the exploitation scheme — only that it's in principle possible to do so. It's the exploitability (rather than any likely exploitation) of your preferential attitudes that is taken to be a sign of irrationality.[16]

It may finally be objected that the uncertainty about what you would choose at nodes 4 and 5 is Knightian (that is, the uncertainty is not quantifiable by exact probabilities). If so, we need a Knightian variation of the Strong Principle of Unidimensional Stochastic Dominance. Let '$[X; Y; \dots]$' denote a Knightian prospect where $X, Y, \dots$ are the potential outcomes that have no quantifiable probabilities. Then we adopt the following requirement of rationality:

> *The Strong Principle of Unidimensional Knightian Dominance* If (i) X^- is a souring of X and $X \succ X^-$, then $[X; Y] \succ [X^-; Y]$.

Since we have no quantifiable probabilities, all we have to work with are the potential outcomes of $[X; Y]$ and the potential outcomes of $[X^-; Y]$, which are the same as those of $[X; Y]$ except that one outcome in the former is soured. So $[X; Y]$ should be preferred to $[X^-; Y]$. Then, letting the probability of E likewise be Knightian, the Strong Principle of Unidimensional Knightian Dominance entails that the prospect of going up at

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prescribed options that terminates the decision problem (options that are not followed by any further choice nodes). See Rabinowicz 1998, p. 101. In BI-terminating decision problems, we can defend the prescription of backward induction at the initial node without making the implausible assumption that one would behave rationally even at choice nodes that are only reachable via irrational choices.

[15] Steele 2010, p. 474 and Gustafsson 2022, p. 14.

[16] See Gustafsson 2022, p. 21.

node 1 is preferred to the prospect (using backward induction) of going down at that node.

But what if your uncertainty about what you would choose at nodes 4 and 5 is imprecise? We can model your imprecise credences as the set of precise probability functions that are sharpenings of your credences. And we adopt the following requirement of rationality:

> *The Strong Principle of Sharpened Unidimensional Stochastic Dominance* If U and V are imprecise prospects such that, on each precise probability function that is a sharpening of the agent's credences, there is a probability p and prospects X, X^-, Y, such that (i) X^- is a souring of X, (ii) $X \succ X^-$, (iii) $0 < p < 1$, U is equivalent to prospect $[X, p; Y, 1-p]$, and V is equivalent to prospect $[X^-, p; Y, 1-p]$, then $U \succ V$.

Then, for each sharpening i of your credences, we let p_i, q_i, and r_i be your credences in i for p, q, and r respectively. Now, we let E be an event that occurs with an imprecise probability r such that, for each sharpening i of your credences, we have

$$(7) \qquad r_i = \frac{p_i}{p_i + q_i}.$$

Then, by the Strong Principle of Sharpened Unidimensional Stochastic Dominance, it follows that the imprecise prospect of going up at node 1 is preferred to the imprecise prospect (using backward induction) of going down at that node. So you would still get exploited. Hence, even with Knightian probabilities, the money-pump argument still works.

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References

Arrow, Kenneth J. (1951) *Social Choice and Individual Values*, New York: Wiley.

Bader, Ralf M. (2019) 'Probabilistic Exploitation', unpublished manuscript.

Bradley, Seamus and Katie Steele (2016) 'Can Free Evidence Be Bad? Value of Information for the Imprecise Probabilist', *Philosophy of Science* 83 (1): 1–28.

Broome, John (1999) *Ethics Out of Economics*, Cambridge: Cambridge University Press.

— (2000) 'Incommensurable Values', in Roger Crisp and Brad Hooker, eds., *Well-Being and Morality: Essays in Honour of James Griffin*, pp. 21–38, Oxford: Clarendon Press.

Buchak, Lara (2013) *Risk and Rationality*, Oxford: Oxford University Press.

Chang, Ruth (1997) 'Introduction', in Ruth Chang, ed., *Incommensurability, Incomparability, and Practical Reason*, pp. 1–34, Cambridge, MA: Harvard University Press.

Gustafsson, Johan E. (2022) *Money-Pump Arguments*, Cambridge: Cambridge University Press.

Gustafsson, Johan E. and Nicolas Espinoza (2010) 'Conflicting Reasons in the Small-Improvement Argument', *The Philosophical Quarterly* 60 (241): 754–763.

Knight, Frank H. (1921) *Risk, Uncertainty and Profit*, Boston: Houghton Mifflin.

McClennen, Edward F. (1990) *Rationality and Dynamic Choice: Foundational Explorations*, Cambridge: Cambridge University Press.

Rabinowicz, Wlodek (1998) 'Grappling with the Centipede: Defence of Backward Induction for BI-Terminating Games', *Economics and Philosophy* 14 (1): 95–126.

Raz, Joseph (1985–1986) 'Value Incommensurability: Some Preliminaries', *Proceedings of the Aristotelian Society* 86:117–134.

— (1986) *The Morality of Freedom*, Oxford: Clarendon Press.

Steele, Katie Siobhan (2010) 'What Are the Minimal Requirements of Rational Choice? Arguments from the Sequential-Decision Setting', *Theory and Decision* 68 (4): 463–487.

Ullmann-Margalit, Edna and Sidney Morgenbesser (1977) 'Picking and Choosing', *Social Research* 44 (4): 757–785.

von Neumann, John and Oskar Morgenstern (1944) *Theory of Games and Economic Behavior*, Princeton: Princeton University Press.