[Published in *Social Choice and Welfare* 35 (1): 65–79, 2010,
DOI: 10.1007/s00355-009-0430-4. The original publication is available at www.springerlink.com.]

Freedom of Choice and Expected Compromise

Johan E. Gustafsson*

This article develops a new measure of freedom of choice based on the proposal that a set offers more freedom of choice than another if, and only if, the expected degree of dissimilarity between a random alternative from the set of possible alternatives and the most similar offered alternative in the set is smaller. Furthermore, a version of this measure is developed, which is able to take into account the values of the possible options.

1. Introduction

In recent years, a growing literature has emerged concerning the measurement of freedom of choice.[1] The aim has been to find an adequate way of ranking sets of options with respect to how much freedom of choice they offer. This article proposes a new measure, the expected-compromise measure. According to this measure, sets of alternatives should be ranked by the expected degree of dissimilarity between a random alternative (from the set of all possible alternatives) and the most similar alternative in each set.

The structure of this article is as follows. Section 2 surveys some of the previous proposals in the literature. In Sect. 3, I develop an unweighted version of the new measure and in Sect. 4 a weighted version. Then, in Sect. 5, some properties of the proposed measure are examined. **[p. 66]**

2. Some previous proposals

In an influential 1990 article, Prasanta K. Pattanaik and Yongsheng Xu present three conditions for ranking sets of alternatives with respect to freedom of choice.[2] Let $\mathbf{\Omega}$ denote the set of all possible alternatives, which we assume to be finite.

> *Indifference between No-choice Situations*: For all $x, y \in \mathbf{\Omega}$, $\{x\}$ offers the same degree of freedom of choice as $\{y\}$.

* I would be grateful for any thoughts or comments on this paper, which can be sent to me at *johan.eric.gustafsson@gmail.com*.

[1] See, e.g. Jones and Sugden (1982), Pattanaik and Xu (1990), Sen (1993), Arrow (1995), Puppe (1996), Pattanaik and Xu (1998), Carter (1999), Pattanaik and Xu (2000), Rosenbaum (2000), Sugden (2003), van Hees (2004), and Bervoets and Gravel (2007).

[2] Pattanaik and Xu (1990, p. 386).

Strict Monotonicity: For all distinct $x, y \in \mathbf{\Omega}$, $\{x, y\}$ offers more freedom of choice than $\{x\}$.

Independence: For all non-empty subsets **U** and **V** and for all $x \in \mathbf{\Omega} \setminus (\mathbf{U} \cup \mathbf{V})$, **U** offers at least as much freedom of choice as **V** iff $\mathbf{U} \cup \{x\}$ offers at least as much freedom of choice as $\mathbf{V} \cup \{x\}$.

Pattanaik and Xu proved that the only measure that satisfies these conditions is the cardinality measure, according to which a set offers at least as much freedom of choice as another iff it has at least as many elements.

A stock objection to the independence condition, and, consequently, to the cardinality measure, is that it does not take into account the degree of dissimilarity between the alternatives.[3] The effects of adding a new alternative that is very similar to one of the original alternatives should be smaller than those of adding a new alternative which is very dissimilar to all the previous ones. However, this would violate the independence condition. The standard way to take dissimilarity into account is to extend the framework with a dissimilarity function. Let $d(x, y)$ denote a function from $\mathbf{\Omega} \times \mathbf{\Omega}$ to $\mathbb{R}_+$ that measures the degree of dissimilarity on a ratio scale, and let it satisfy $d(x, y) = d(y, x)$, $d(x, x) = 0$. Moreover we assume that if $d(x, y) = 0$ then $x = y$.[4]

For an illustration, suppose that you are to choose a temperature for your office. The possible alternatives are $\mathbf{\Omega} = \{0\ °C, 1\ °C, 2\ °C, \ldots, 30\ °C\}$. In this case, a natural dissimilarity function on $\mathbf{\Omega} \times \mathbf{\Omega}$ is simply the difference in temperature between the alternatives. For all $x, y \in \mathbf{\Omega}$, $d(x, y) = |\text{temp}(x) - \text{temp}(y)|$. In Figure 1 the alternatives in the subsets **A** and **B** of $\mathbf{\Omega}$ are visualized as points on a line from 0 to 30.

$\mathbf{A} = \{0\ °C, 1\ °C\}$

$\mathbf{B} = \{0\ °C, 30\ °C\}$

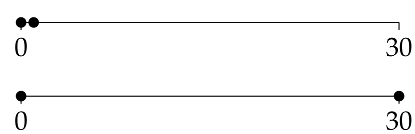

Figure 1: The sets **A** and **B**.

Suppose you judge, in line with the indifference between no-choice situations condition, that $\{1\ °C\}$ offers an equal amount of freedom of choice as $\{30\ °C\}$. In order to satisfy the independence condition, you also have to judge that **A** and **B** offer equal amounts of freedom of choice. However, intuitively **B** seems to offer more freedom of choice than **A** due to the low degree of dissimilarity between the alternatives in **A**. Since both **A** and **B** have the same number of elements, they offer the same amount of freedom of choice according to the cardinality measure. It seems, therefore, that the cardinality measure does not fit our intuitions.

[3]Pattanaik and Xu (1990, p. 390).

[4]A standard assumption, see, e.g. van Hees (2004, p. 257), is that the dissimilarity function satisfies the triangle inequality, $d(x, z) \leq d(x, y) + d(y, z)$. This assumption is not obviously valid, and is not needed for my purposes.

A number of measures that take dissimilarity into account have been proposed. For example, Eckehard F. Rosenbaum proposed the maximum-dissimilarity measure **[p. 67]** according to which a set **U** offers at least as much freedom of choice as a set **V** iff the greatest distance between the two alternatives in **U** is at least as great as the greatest distance between two alternatives in **V**.[5] According to this measure, **B** would offer more freedom of choice than **A**, since the maximum dissimilarity between two elements is 30 in **B** but just 1 in **A**. This seems to be in accordance with our intuitions.

However, what happens if we add a third alternative, 15 °C, to **B** as in the set **C** in Fig. 2? Since **C** in addition to the cold and hot alternatives in **B** also

$\mathbf{C} = \{0\,°C, 15\,°C, 30\,°C\}$

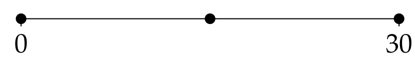

Figure 2: The set **C**.

offers an alternative from the middle range that was unrepresented in **B**, it intuitively offers more freedom of choice than **B**. However, according to the maximum-dissimilarity measure **B** and **C** offer just as much freedom of choice. Again, this is not in accordance with intuition.

In order to take diversity into account, Pattanaik and Xu have proposed what they call 'the simple similarity based ordering', according to which sets of alternatives are ranked with respect to freedom of choice by the cardinality of the smallest similarity-based partition of the sets.[6] A problem with this approach is that it does not distinguish between different degrees of dissimilarity. A counter-intuitive result of this is that any three sets with two elements cannot all offer a different degree of freedom of choice, since there are just two possible cardinalities of a partitioning of a set with two elements. For example, the sets $\{\textit{blue train}, \textit{red train}\}$, $\{\textit{blue train}, \textit{red bus}\}$, and $\{\textit{blue train}, \textit{red wine}\}$ cannot all offer different degrees of freedom of choice. Martin van Hees has tried to reformulate Pattanaik and Xu's framework taken into account the degrees of dissimilarity between alternatives, but with negative results.[7]

Another criticism of the cardinality measure is that preferences need to be taken into account. In order to take preferences into account, Pattanaik and Xu have proposed a variation of their cardinality measure that also measures the number of alternatives in a set of alternatives, but only those which are considered best in that set by a reasonable person.[8] However, the measure inherits some of the problems of the cardinality measure, since it also does not take the degree of similarity between alternatives into account. For example, if all alternatives in **A** and **B** are considered best in the set by a reasonable person, then Pattanaik and Xu's preference sensitive measure, like the **[p. 68]** cardinality measure, implies that **A** offers the same amount of freedom of

[5] Rosenbaum (2000, p. 216).
[6] Pattanaik and Xu (2000, p. 127).
[7] van Hees (2004).
[8] Pattanaik and Xu (1998, p. 187).

choice as **B**. But this again is counter-intuitive.

Sebastian Bervoets and Nicolas Gravel use a similar approach, but they also take the diversity of the alternatives in the sets into account. Their measure ranks sets of alternatives in order of the maximum degree of dissimilarity between two alternatives such that each is considered best in that set by a reasonable person.[9] A virtue of their measure is that it, unlike the expected-compromise measure, only needs an ordinal dissimilarity function. However, it also has counter-intuitive features. The measure is very similar to Rosenbaum's maximum-dissimilarity measure, and has similar problems. As an example, suppose that all the alternatives in **B** and **C** are considered best in the respective set by a reasonable person. Then, Bervoets and Gravel's measure implies, like the maximum-dissimilarity measure, that **B** offers the same amount of freedom of choice as **C**. As before, this is contrary to intuition.

Kenneth Arrow and Clemens Puppe have proposed measures of freedom of choice based on preference for flexibility.[10] To have a preference for flexibility is roughly to prefer to have a wide range of options at a later time.[11] Arrow ranks sets of alternatives with respect to freedom of choice by the expected utility of the sets given a probability distribution over utility functions. This is a promising approach, although more need to be said about which probability distribution should be used. I will develop this approach in Sect. 3.

3. The expected-compromise measure

Neither of the measures discussed above (except perhaps the flexibility approach) properly accounts for the intuition that freedom of choice increases as alternatives from a new range of the set of possible alternatives become available. I will now develop a proposal that solves this problem.

Consider

> *The Unpredictable Boss*. Suppose you are going to prepare a set of alternatives from which your boss will choose one. You know that the boss has a favourite alternative in the set of all possible alternatives, and that he wants to choose an alternative that is as similar as possible to his favourite alternative. You estimate that all possible alternatives have the same probability of being the boss's favourite alternative.
>
> Suppose you have to choose between offering the boss one of two different sets of alternatives, **D** and **E**, and you happen to know that **D** offers more freedom of choice than **E**. If you want to minimize the expected degree of dissimilarity between the boss's favourite alternative and the least dissimilar alternative in the set of

[9] Bervoets and Gravel (2007, p. 268).
[10] Arrow (1995), Puppe (1996).
[11] See, e.g. Koopmans (1964), Kreps (1979).

> alternatives you offer him, which of the sets, **D** and **E**, would you offer him?

[p. 69] My intuition is that you should offer the boss **D**, the set that offers him more freedom of choice. It seems plausible that an agent with a random favourite alternative is more likely to be able to choose an alternative more similar to his favourite from a set that offers much freedom of choice than a set that offers little.

Here is another example: Suppose that each day your tutor picks, at random, a fruit from a giant bowl, containing one of each possible fruit. Each day it is your job to find another fruit as similar as possible to the one your tutor picked. It seems plausible that, in general, you would find a fruit more similar to your tutor's randomly picked fruit, if you each day had to choose from a store whose selection offered much freedom of choice than from a store whose selection offered little. The expected compromise you have to make when choosing a fruit would be smaller when the selection offers more freedom of choice.

The connection between freedom of choice and the expected degree of dissimilarity (between a random possible alternative and the most similar offered alternative) suggests the following measure of freedom of choice:

> *The Expected-Compromise Measure*: Given the domain $\mathbf{\Omega}$, **U** offers at least as much freedom of choice as **V** iff the expected degree of dissimilarity between a random alternative from $\mathbf{\Omega}$ and the least dissimilar alternative in **U** is at least as low as the expected degree of dissimilarity between a random alternative from $\mathbf{\Omega}$ and the least dissimilar alternative in **V**.

I shall now develop a more precise version of this proposal. For simplicity, we start from the assumption that it is equally probable that each alternative in $\mathbf{\Omega}$ is picked. Other probability distributions will be discussed in Sect. 4. Let the picked alternative be the value of the random variable X. We then have the following probability function:

$$p_X(x) = P(X = x) = \frac{1}{|\mathbf{\Omega}|}.$$

Let $D(x, \mathbf{U})$ denote the minimal dissimilarity between the element x and an element in **U**, $D(x, \mathbf{U}) = \min(\{d(x, y) : y \in \mathbf{U}\})$. The expected degree of dissimilarity between this random alternative and the least dissimilar alternative in **U** is

$$E(D(X, \mathbf{U})) = \sum_{x \in \mathbf{\Omega}} D(x, \mathbf{U}) p_X(x) = \frac{\sum_{x \in \mathbf{\Omega}} D(x, \mathbf{U})}{|\mathbf{\Omega}|}.$$

This allows us to state a more precise version of the expected-compromise measure:

The Unweighted Expected-Compromise Measure: Given the domain $\mathbf{\Omega}$, the non-empty subset **U** offers at least as much freedom of choice as the non-empty subset **V** iff

$$\sum_{x \in \mathbf{\Omega}} D(x, \mathbf{U}) \leq \sum_{x \in \mathbf{\Omega}} D(x, \mathbf{V}).$$

[p. 70] According to this measure **U** offers more freedom of choice than **V** if, and only if, the sum of the minimal degrees of dissimilarity to an alternative in **U** for each of all the possible alternatives is smaller than the sum of the minimal degrees of dissimilarity to an alternative in **V** for each of all the possible alternatives.

A feature of the measure that initially might seem strange is that it uses the entire domain, $\mathbf{\Omega}$, as a reference set. For most measures of freedom of choice, the only alternatives relevant for a comparison are those in the compared sets. Why does it make sense to use $\mathbf{\Omega}$ as a reference set? Note first that $\mathbf{\Omega}$ has a special significance for rankings of sets in terms of freedom; any plausible measure of freedom of choice should rank the whole domain as offering at least as much freedom of choice as any subset of the domain. So $\mathbf{\Omega}$ represents an ideal choice set with respect to freedom of choice, since any alternative can be chosen and any preference can be expressed. As the subsets of $\mathbf{\Omega}$ get smaller, there are more possible preferences for which one has to settle for an alternative dissimilar to the optimal. These possible compromises should intuitively decrease the amount of freedom of choice offered. Therefore, I think it makes sense to rank sets by some measure of similarity to $\mathbf{\Omega}$, like the expected-compromise measure.

The expected-compromise measure can also be obtained with Arrow's approach as a starting point, with a few further plausible assumptions on the probabilities of the possible utility functions. Arrow ranks sets of alternatives with respect to freedom of choice with the following *freedom evaluation function*:[12]

$$V(\mathbf{A}) = E_\theta[\max(\{U(x, \theta) : x \in \mathbf{A}\})],$$

where $U(x, \theta)$ is the utility of the alternative x given the random parameter θ and $E_\theta[f(\theta)]$ is the expected value of $f(\theta)$.

I take the underlying intuition behind this approach to be roughly that freedom of choice increases, if one is able to better satisfy a set of relevant possible preferences. However, the approach needs to be complemented with an account of which are the relevant preferences or utility functions. One possible answer is to use a set of preferences that prefers one alternative over all others and holds the other alternatives to be equally bad, and to let a preference for each of the possible alternatives be equally probable:

$$\forall x \in \mathbf{\Omega}, P(\theta = x) = \frac{1}{|\mathbf{\Omega}|} \tag{1}$$

[12] Arrow (1995, pp. 10–11).

$$U(x,\theta) = \begin{cases} 1 & \text{if } \theta = x \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Given (1) and (2) Arrow's function will rank sets according to the cardinality measure. Similar to the cardinality measure, this probability distribution is problematic as it does not take into account the degree of similarity between possible alternatives. For each utility function with a positive probability, we have one optimal alternative, and all the other alternatives, however similar to the optimal alternative, are equally bad. It seems that a more plausible utility function would be to have an optimal alternative **[p. 71]** and then prefer other alternatives gradually less as they become more dissimilar to the optimal alternative. In order to obtain utility functions like that we can replace (2) with the following:

$$U(x,\theta) = -d(x,\theta) \tag{2'}$$

If we supplement Arrow's approach with (1) and (2′) instead of (1) and (2), then Arrow's function coincides with the unweighted expected-compromise measure.

In order to see how the expected-compromise measure works, we will return to the temperature example. In order to compare the sets of alternatives $\mathbf{A} = \{0\ ^\circ\mathrm{C}, 1\ ^\circ\mathrm{C}\}$ and $\mathbf{B} = \{0\ ^\circ\mathrm{C}, 30\ ^\circ\mathrm{C}\}$, we have to consider the degree of dissimilarity between each alternative in the set of all possible alternatives $\mathbf{\Omega} = \{0\ ^\circ\mathrm{C}, 1\ ^\circ\mathrm{C}, 2\ ^\circ\mathrm{C}, \ldots, 30\ ^\circ\mathrm{C}\}$ and the least dissimilar alternative in **A** and **B**.

In Fig. 3 the value of $D(x, \mathbf{U})$ for each possible alternative has been added to the sets from Figs. 1 and 2, that returns the degree of dissimilarity between an alternative x and the least dissimilar in the set of alternatives **U**. The values of $D(x, \mathbf{U})$ have been added as bars to the line, which represented the set of all possible alternatives. Thus, the height of a bar over an alternative represents the degree of dissimilarity between the alternative and the least dissimilar alternative in the set of alternatives. In Figure 3, the bars in **A** get taller to the right as they get further from the two offered alternatives to the left. In **B** the bars are tallest in the middle around 15 °C and get shorter as the alternatives get more similar to one of the offered alternatives to the far left and right. Hence, the expected-compromise measure has an intuitive graphical interpretation. The freedom of choice in a set of alternatives increases as the sum of the heights of all the bars decreases.

Since the sum of $D(x, \mathbf{B})$ for all possible alternatives x is smaller than the sum of $D(x, \mathbf{A})$ for all possible alternatives x, **B** offers more freedom of choice than **A**.[13] As mentioned in Sect. 2, the cardinality measure has the disadvantage of not reflecting that the alternatives in **A** came from a very similar range of

[13] Since the unweighted expected-compromise measure implies that $\{0\ ^\circ\mathrm{C}\}$ offers at least as much freedom of choice as $\{30\ ^\circ\mathrm{C}\}$ and also that $\{0\ ^\circ\mathrm{C}\} \cup \{1\ ^\circ\mathrm{C}\}$ do not offer at least as much freedom of choice as $\{30\ ^\circ\mathrm{C}\} \cup \{1\ ^\circ\mathrm{C}\}$, it violates the independence condition.

$\mathbf{A} = \{0\ ^\circ\mathrm{C}, 1\ ^\circ\mathrm{C}\}$

$\mathbf{B} = \{0\ ^\circ\mathrm{C}, 30\ ^\circ\mathrm{C}\}$

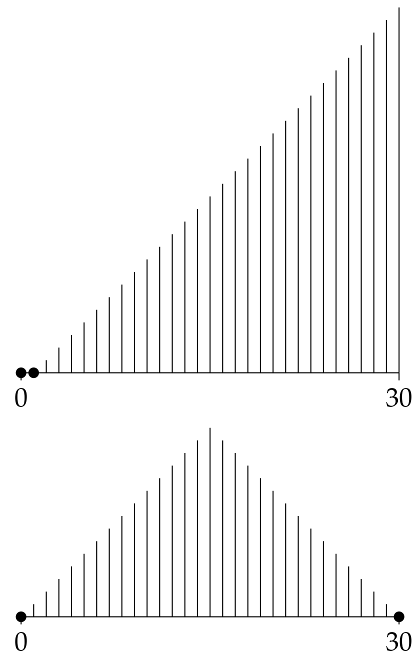

Figure 3: The dissimilarity with the most similar alternative in **A** and **B** for all possible alternatives.

the set of all possible alternatives, whereas the alternatives in **B** came from two very dissimilar ranges of the set of all possible alternatives. This is reflected in the expected-compromise measure. Since **A** has no warm alternatives, the degree of dissimilarity to the least dissimilar alternative in **A** becomes very high for the warm range of the set of all possible alternatives.

Next, let us turn to the comparison of **B** with **C** that was troublesome for the maximum-dissimilarity measure. As seen in Fig. 4 the new alternative, 15 °C,

$\mathbf{C} = \{0\ ^\circ\mathrm{C}, 15\ ^\circ\mathrm{C}, 30\ ^\circ\mathrm{C}\}$

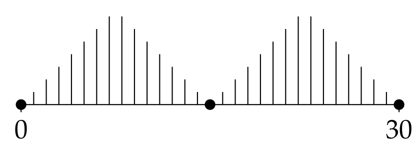

Figure 4: The dissimilarity with the most similar alternative in **C** for all possible alternatives.

makes the degree of dissimilarity smaller than in **B** for all possible alternatives between 7.5 °C and 22.5 °C and the same as in **B** for all other alternatives. Thus, the sum of $D(x, \mathbf{C})$ for all possible alternatives x is smaller than the sum of $D(x, \mathbf{B})$ for all possible alternatives x. Therefore **C** offers more freedom of choice than **B** according to the expected-compromise measure. This fits with our intuitions.

[p. 72] The problem for the maximum-dissimilarity measure was that it did not reflect that the amount of freedom of choice offered to an agent seems to increase when alternatives from a new range of the set of possible alternatives become available to the agent. This is reflected by the expected-compromise measure.

4. A weighted version of the measure

In addition to the failure to account for diversity, there has been another line of response against the cardinality measure. Amartya Sen argues that it is counter-intuitive that a set of three alternatives that are seen as 'bad', 'terrible' and 'disastrous' offers as much freedom of choice as a set of three alternatives that are seen as 'good', 'terrific' and 'wonderful'.[14] In other words, the intuition is that an adequate measure should take into account that good alternatives are more relevant for an agent's freedom of choice than bad alternatives. The measure proposed, in this article can easily be modified to take Sen's very plausible intuition into account.

[p. 73] In the unweighted version of the expected-compromise measure, all possible alternatives are assumed to be equally relevant for the degree of freedom of choice that a set of alternatives offers. In light of Sen's intuition, this might not seem plausible. In response to this, I will introduce a weight function that gives each possible alternative a weight. Let $w(x)$ denote a function on $\mathbf{\Omega}$ that returns a weight on a ratio scale for each possible alternative. These weights could represent the value or relevance of each possible alternative. For example, consider again the example with sets of alternative temperatures for an office.

In this case, it seems strange that each of the possible alternatives in $\mathbf{\Omega} = \{0\,^\circ\mathrm{C}, 1\,^\circ\mathrm{C}, 2\,^\circ\mathrm{C}, \ldots, 30\,^\circ\mathrm{C}\}$ would be equally relevant to the boss's freedom of choice. Who would want 0 °C in their office? 0 °C appears to be much less relevant than 20 °C. A simple weight function for this case that gives more weight to the warmer alternatives than the cold ones is, $w(x) = 20 - |\mathrm{temp}(x) - 20|$.

Instead of assigning equal probability to all possible alternatives of being the boss's favourite, we assign to each possible alternative a probability as follows:

$$p_X(x) = \frac{w(x)}{\sum_{y \in \mathbf{\Omega}} w(y)}.$$

With these probabilities, we get a weighted version of the expected-compromise measure:

> *The Weighted Expected-Compromise Measure*: Given the domain $\mathbf{\Omega}$, the non-empty subset $\mathbf{U}$ offers at least as much freedom of choice as the non-empty subset $\mathbf{V}$ iff
>
> $$\sum_{x \in \mathbf{\Omega}} w(x) D(x, \mathbf{U}) \leq \sum_{x \in \mathbf{\Omega}} w(x) D(x, \mathbf{V}).$$

[14] See Sen (1990, p. 470) and Sen (1993, p. 529).

This measure differs from the unweighted version only in that the dissimilarity of the alternatives in a set to a possible alternative is multiplied by the weight of that possible alternative. This implies that a set of alternatives will not be heavily penalized for not covering a range of alternatives with low weights, such as in our example, the range of extremely cold alternatives no one would want for their office.

5. Properties of the measure

In this section, I discuss some formal properties of the expected-compromise measure, viz. localized independence, domain sensitivity, monotonicity, and moderation. I shall also examine the widely discussed indifference between no-choice situations condition and the measure's relation to the multi-attribute approach.

Pattanaik and Xu's independence condition has been criticized for not taking similarity into account, as was mentioned in Sect. 2. The expected-compromise measure violates independence, but it satisfies a related similarity sensitive condition: **[p. 74]**

> *Localized Independence*: For all non-empty subsets $\mathbf{U}$ and $\mathbf{V}$ and for all $x \in \mathbf{\Omega}$, if for all $z \in \mathbf{\Omega}$, $[d(z,x) < \max(\{D(z,\mathbf{U}), D(z,\mathbf{V})\})] \rightarrow [D(z,\mathbf{U}) = D(z,\mathbf{V})]$ then $\mathbf{U}$ offers at least as much freedom of choice as $\mathbf{V}$ iff $\mathbf{U} \cup \{x\}$ offers at least as much freedom of choice as $\mathbf{V} \cup \{x\}$.[15]

Pattanaik and Xu's independence was problematic, since it required that the ranking of $\mathbf{U}$ and $\mathbf{V}$ should be the same as that for $\mathbf{U} \cup \{x\}$ and $\mathbf{V} \cup \{x\}$ even if x is very similar to an alternative in $\mathbf{U}$ but dissimilar to all alternatives in $\mathbf{V}$. This problem does not affect localized independence, since it only requires that extensions of $\mathbf{U}$ and $\mathbf{V}$ with options from a range that is equally well-represented in $\mathbf{U}$ as in $\mathbf{V}$ should be ranked the same.

In order to see that the weighted expected-compromise measure satisfies localized independence, note that it ranks sets $\mathbf{U}$ by sum of $w(y)D(y,\mathbf{U})$ for all alternatives y in $\mathbf{\Omega}$. For all $z \in \mathbf{\Omega}$ such that $d(z,x) \geq \max(\{D(z,\mathbf{U}), D(z,\mathbf{V})\})$ we have that $w(z)D(z,\mathbf{U}) = w(z)D(z,\mathbf{U} \cup \{x\})$ and that $w(z)D(z,\mathbf{V}) = w(z)D(z,\mathbf{V} \cup \{x\})$; so these will be unaffected by the extension. For all $z \in \mathbf{\Omega}$ such that $d(z,x) < \max(\{D(z,\mathbf{U}), D(z,\mathbf{V})\})$ and $D(z,\mathbf{U}) = D(z,\mathbf{V})$ we have that $w(z)D(z,\mathbf{U}) = w(z)D(z,\mathbf{V})$ and that $w(z)D(z,\mathbf{U} \cup \{x\}) = w(z)D(z,\mathbf{V} \cup \{x\})$; so even though these will be affected by the extension of $\{x\}$ it will be by the same amount for $\mathbf{U}$ and $\mathbf{V}$. Thus, if for all $z \in \mathbf{\Omega}$, $[d(z,x) < \max(\{D(z,\mathbf{U}), D(z,\mathbf{V})\})] \rightarrow [D(z,\mathbf{U}) = D(z,\mathbf{V})]$, the weighted measure ranks $\mathbf{U}$ and $\mathbf{V}$ the same as $\mathbf{U} \cup \{x\}$ and $\mathbf{V} \cup \{x\}$, which was to be shown. Since the unweighted measure, is just a special case of the weighted measure it follows that the former also satisfies localized independence.

[15] In this article '$\rightarrow$' denotes a material implication.

Van Hees has argued that freedom is sensitive to variations of the relevant domain or the set of technologically feasible options.[16] The expected-compromise measure differs from most other dissimilarity-based measures in that it is sensitive to what is included in the domain of possible alternatives, and not just to the alternatives in the compared sets of alternatives. It satisfies the following condition:

> *Domain Sensitivity*: There exist domains $\mathbf{\Omega}'$ and $\mathbf{\Omega}''$ and non-empty sets of alternatives $\mathbf{X}$ and $\mathbf{Y}$ such that $\mathbf{X}, \mathbf{Y} \subseteq \mathbf{\Omega}'$, $\mathbf{X}, \mathbf{Y} \subseteq \mathbf{\Omega}''$, and $\mathbf{X}$ offers at least as much freedom of choice as $\mathbf{Y}$ given domain $\mathbf{\Omega}'$ and $\mathbf{X}$ does not offer at least as much freedom of choice as $\mathbf{Y}$ given domain $\mathbf{\Omega}''$.

For an example of domain sensitivity, let $\mathbf{\Omega}_1 = \{10\,°\text{C}, 11\,°\text{C}, 12\,°\text{C}, \ldots, 20\,°\text{C}\}$. Given that the domain of possible alternatives is $\mathbf{\Omega}_1$, consider the two sets of alternatives in Fig. 5. In this case, I think it is intuitively reasonable to judge

$\mathbf{F} = \{10\,°\text{C}, 20\,°\text{C}\}$

$\mathbf{G} = \{12\,°\text{C}, 18\,°\text{C}\}$

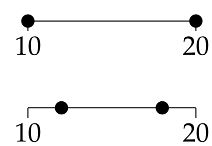

Figure 5: Sets **F** and **G** given domain $\mathbf{\Omega}_1$.

that **G** offers more freedom of choice than **F**, as the unweighted expected-compromise measure does. The alternatives in **G** are less extreme relative to the domain than those in **F** and are arguably more representative of the range of possible choices from $\mathbf{\Omega}_1$.

Now, we will compare **F** and **G** again but with another domain. Let $\mathbf{\Omega}_2 = \{0\,°\text{C}, 1\,°\text{C}, 2\,°\text{C}, \ldots, 30\,°\text{C}\}$. Consider **F** and **G** in Fig. 6 with the larger domain $\mathbf{\Omega}_2$. This time it seems more intuitively plausible that **F** offers more freedom

$\mathbf{F} = \{10\,°\text{C}, 20\,°\text{C}\}$

$\mathbf{G} = \{12\,°\text{C}, 18\,°\text{C}\}$

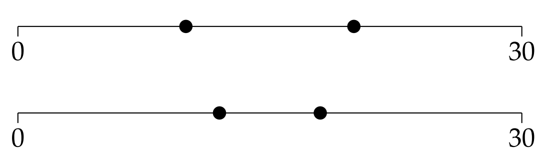

Figure 6: Sets **F** and **G** given domain $\mathbf{\Omega}_2$.

of choice than **G**. Given the new domain, the alternatives in **F** are no longer extreme. The new **[p. 75]** extended range of alternatives is better represented by the alternatives in **F** which compensates the better coverage of the middle range in **G**. This intuition also fits with the unweighted expected-compromise measure as it, with domain $\mathbf{\Omega}_2$, implies that **F** offers more freedom of choice than **G**. This shows that the expected-compromise measure satisfies domain sensitivity.

[16] van Hees (1998).

It is easy to show that the unweighted expected-compromise measure satisfies the strict monotonicity condition, presented in Sect. 2. Of Pattanaik and Xu's three conditions, this is the least controversial one. It can also trivially be shown that the unweighted expected-compromise measure satisfies the following monotonicity condition:

> *Strong Monotonicity*: For all non-empty sets of alternatives $\mathbf{U} \subset \mathbf{\Omega}$ and for all alternatives $x \in \mathbf{\Omega}$ such that $w(x) > 0$ and $d(x, y) > 0$ for all $y \in \mathbf{U}$, $\mathbf{U} \cup \{x\}$ offers more freedom of choice than $\mathbf{U}$.

The weighted expected-compromise measure satisfies strong monotonicity, but not strict monotonicity, since if an added alternative does not have a positive weight, it might not increase the degree of freedom of choice.

A possible objection to the expected-compromise measure is that it does not satisfy the indifference between no-choice situations condition (INS), according to which all singleton sets of alternatives always offer the same degree of freedom of choice.[17] The intuition given in support of INS in the literature is that all singleton sets offer *no* freedom of choice and therefore they offer the same degree of freedom of choice.[18]

The INS condition is not, however, without critics. Sen has argued that it is absurd to say that the sets {hop on one leg to home} and {walk normally home} offer exactly as much freedom of choice.[19] Singleton sets with an alternative, we would not have chosen if we had the option to avoid it do not offer as much 'freedom to live as we would like', as singleton sets with an alternative that we would have chosen even if **[p. 76]** we could have had any possible alternative.[20] Furthermore, Peter Jones and Robert Sugden have shown that INS is incompatible with the conjunction of the following plausible principles: if a choice set is extended by an alternative that is a *significant* addition to it, then the freedom of choice offered increases, and if a choice set is extended by an alternative that is an *insignificant* addition to it, then the freedom of choice offered does not increase.[21]

Like Sen I believe that the INS condition should be dropped since it conflicts with more compelling intuitions.[22] Rather than INS, the expected-compromise measure ranks singleton sets based on a conflicting moderation intuition. The

[17] This condition was first proposed by Peter Jones and Robert Sugden, Jones and Sugden (1982, p. 56).

[18] Jones and Sugden (1982, p. 56), Pattanaik and Xu (1990, pp. 386–387), Pattanaik and Xu (1998, p. 183).

[19] Sen (1990, p. 471).

[20] Sen (1991, p. 25).

[21] See Jones and Sugden (1982, p. 57) and Sugden (1998, p. 328).

[22] If one still is unwilling to give up INS, one could make my approach compatible with INS by a slight modification:

> *The INS-Expected-Compromise Measure*: Given the domain $\mathbf{\Omega}$, the non-empty subset $\mathbf{U}$ offers at least as much freedom of choice as the non-empty subset $\mathbf{V}$ iff
>
> (1) $\mathbf{V}$ is a singleton set, *or*
>
> (2) neither $\mathbf{U}$ nor $\mathbf{V}$ is a singleton set, and $\sum_{x \in \mathbf{\Omega}} w(x) D(x, \mathbf{U}) \leq \sum_{x \in \mathbf{\Omega}} w(x) D(x, \mathbf{V})$.

unweighted expected-compromise measure satisfies the following new condition:

> *Moderation*: For all singleton sets $\{x\}$ and $\{y\}$ such that $x, y \in \mathbf{\Omega}$, $\{x\}$ offers at least as much freedom of choice as $\{y\}$ iff
>
> $$\sum_{z \in \mathbf{\Omega}} d(x, z) \leq \sum_{z \in \mathbf{\Omega}} d(y, z).$$

The intuition behind the condition is that singleton sets with extreme alternatives offer less freedom of choice than singleton sets with more moderate alternatives in the domain. The moderate sets offer more freedom of choice than the extreme sets, not because they offer more alternatives but because they offer a more similar alternative to more possible alternatives, than the extreme sets do. For example, the set {vote center party} arguably offers more freedom of choice than {vote extreme right} or {vote extreme left}, since, in the middle of the political spectrum, it offers a more similar alternative to more possible political views than the extreme sets do.

The weighted expected-compromise measure satisfies a weighted version of the moderation condition:

> *Weighted Moderation*: For all singleton sets $\{x\}$ and $\{y\}$, such that $x, y \in \mathbf{\Omega}$, $\{x\}$ offers at least as much freedom of choice as $\{y\}$ iff
>
> $$\sum_{z \in \mathbf{\Omega}} d(x, z) w(z) \leq \sum_{z \in \mathbf{\Omega}} d(y, z) w(z).$$

The intuition behind the weighted moderation condition is the same as the one behind the unweighted version, with the exception that it is more important for freedom of **[p. 77]** choice that a set offers an alternative that is similar to possible alternatives with greater weights.

Note that the moderation conditions do not rule out that there are domains, in which any two singleton sets offer the same degree of freedom of choice. Suppose that you have to choose a direction and the set of all the possible alternatives was $\mathbf{\Omega} = \{0\,^\circ, 1\,^\circ, 2\,^\circ, \ldots, 359\,^\circ\}$. A natural dissimilarity function on $\mathbf{\Omega} \times \mathbf{\Omega}$ would in this case, be the minimal angle between the directions. For all $x, y \in \mathbf{\Omega}$, $d(x, y) = \min(\{|\deg(x) - \deg(y)|, 360 - |\deg(x) - \deg(y)|\})$. Suppose also that all directions have the same weight. In this case, no alternative is more extreme or moderate than the others. Therefore, according to either of the moderation conditions, no singleton set offers more freedom of choice than any other, which is very intuitive.

In the related literature on how to measure the diversity offered by a set, Claus Nehring and Clemens Puppe have developed the very influential multi-attribute approach. This approach ranks sets of alternatives by their diversity value, which is the sum of all the weights for all the attributes realized by the set. In Nehring and Puppe's framework, an attribute is any subset $\mathbf{A} \subseteq \mathbf{\Omega}$. An attribute $\mathbf{A}$ is realized by a set, if there is an element in the set that is also an element in **A**. The relative importance or weight of an attribute **A** is denoted by '$\lambda_{\mathbf{A}}$'. The diversity value of a set **U** is then defined as follows: [23]

$$v(\mathbf{U}) = \sum_{\mathbf{A} \subseteq \mathbf{\Omega} : \mathbf{A} \cap \mathbf{U} \neq \emptyset} \lambda_{\mathbf{A}}$$

I will now show that the expected-compromise measure is equivalent to a version of the multi-attribute approach. That is, with certain constraints on the attribute weights, the multi-attribute approach and the expected-compromise measure rank sets of alternatives equivalently.

The weighted expected-compromise measure ranks sets of alternatives according to the following function:

$$e(\mathbf{U}) = -\sum_{x \in \mathbf{\Omega}} w(x) D(x, \mathbf{U})$$

The set of alternatives that offers the maximum freedom of choice according to the expected-compromise measure is the set of all possible alternatives, i.e. $\mathbf{\Omega}$. If $\mathbf{\Omega}$ is offered, we have that $e(\mathbf{\Omega}) = 0$ and that all possible features are realized. If a set of alternatives **A** is removed, we have that $e(\mathbf{\Omega}) - e(\mathbf{\Omega} \setminus \mathbf{A}) = \sum_{x \in \mathbf{\Omega}} w(x) D(x, \mathbf{\Omega} \setminus \mathbf{A})$, and that $v(\mathbf{\Omega}) - v(\mathbf{\Omega} \setminus \mathbf{A}) = \sum_{\mathbf{S} \subseteq \mathbf{A} : \mathbf{S} \neq \emptyset} \lambda_{\mathbf{S}}$. Thus the functions $v(\cdot)$ and $e(\cdot)$ give equivalent rankings if for all sets $\mathbf{A} \subset \mathbf{\Omega}$ such that $\mathbf{A} \neq \emptyset$,

$$\sum_{x \in \mathbf{\Omega}} w(x) D(x, \mathbf{\Omega} \setminus \mathbf{A}) = \sum_{\mathbf{S} \subseteq \mathbf{A} : \mathbf{S} \neq \emptyset} \lambda_{\mathbf{S}}. \quad (3)$$

[p. 78] This provides us with a formula for how to assign attribute weights in order for $v(\cdot)$ and $e(\cdot)$ to give equivalent rankings. First, we let $\lambda_{\emptyset} = \lambda_{\mathbf{\Omega}} = 0$.[24] It follows from (3) that for all sets $\mathbf{A} \subset \mathbf{\Omega}$ such that $\mathbf{A} \neq \emptyset$,

$$\lambda_{\mathbf{A}} = \sum_{z \in \mathbf{\Omega}} w(z) D(z, \mathbf{\Omega} \setminus \mathbf{A}) - \sum_{\mathbf{S} \subset \mathbf{A} : \mathbf{S} \neq \emptyset} \lambda_{\mathbf{S}}. \quad (4)$$

For all singleton sets **U** we have that $\lambda_{\mathbf{U}} = \sum_{z \in \mathbf{\Omega}} w(z) D(z, \mathbf{\Omega} \setminus \mathbf{U})$, and the attribute weights for all larger sets follow recursively from (4).

We have now assigned attribute weights in such a way that the multi-attribute approach and the expected-compromise measure rank sets of alternatives equivalently. The weights have been assigned so that the attribute weights for all singletons are the degree of dissimilarity between the sets' element to the most similar of all the other options in the domain multiplied by the element's weight. Larger sets have been assigned attribute weights that are the sum of the degree of dissimilarity between each of the sets' elements to the most similar of all the options in the domain not included in the set multiplied by the element's weight minus the attribute weights for all the subsets. Any positive linear transformation of the attribute weights passing through the origin would, of course, lead to the same result. The question whether this

[23] Nehring and Puppe (2002, p. 1161).

[24] These weights are arbitrary and do not affect the rankings.

version of the multi-attribute approach is a plausible measure of diversity is left for future work.

I wish to thank John Cantwell, Erik Carlson, Karin Enflo, Nicolas Espinoza, Sven Ove Hansson, Fredrik Johansson, Martin Peterson, Tor Sandqvist, and two anonymous referees for valuable comments on earlier versions.

References

Arrow, Kenneth J. (1995) 'A Note on Freedom and Flexibility', in Kaushik Basu, Prasanta K. Pattanaik, and Kotaro Suzumura, eds., *Choice, Welfare and Development: A Festschrift in Honour of Amartya K. Sen*, pp. 7–15, Oxford: Clarendon Press.

Bervoets, Sebastian and Nicolas Gravel (2007) 'Appraising Diversity with an Ordinal Notion of Similarity: An Axiomatic Approach', *Mathematical Social Sciences* 53 (3): 259–273.

Carter, Ian (1999) *A Measure of Freedom*, Oxford: Oxford University Press.

Jones, Peter and Robert Sugden (1982) 'Evaluating Choice', *International Review of Law and Economics* 2 (1): 47–65.

Koopmans, Tjalling C. (1964) 'On Flexibility of Future Preference', in Maynard W. Shelly and Glenn L. Bryan, eds., *Human Judgments & Optimality*, pp. 243–254, New York: Wiley.

Kreps, David M (1979) 'A Representation Theorem for "Preference for Flexibility"', *Econometrica* 47 (3): 565–77.

Nehring, Klaus and Clemens Puppe (2002) 'A Theory of Diversity', *Econometrica* 70 (3): 1155–1198.

Pattanaik, Prasanta K. and Yongsheng Xu (1990) 'On Ranking Opportunity Sets in Terms of Freedom of Choice', *Recherches Economiques de Louvain* 56 (3-4): 383–390.

— (1998) 'On Preference and Freedom', *Theory and Decision* 44 (2): 173–198.

— (2000) 'On Diversity and Freedom of Choice', *Mathematical Social Sciences* 40 (2): 123–130.

Puppe, Clemens (1996) 'An Axiomatic Approach to "Preference for Freedom of Choice"', *Journal of Economic Theory* 68 (1): 174–199.

Rosenbaum, Eckehard F. (2000) 'On Measuring Freedom', *Journal of Theoretical Politics* 12 (2): 205–227.

Sen, Amartya (1990) 'Welfare, Freedom and Social Choice: A Reply', *Recherches Economiques de Louvain* 56 (3-4): 451–485.

— (1991) 'Welfare, Preference and Freedom', *Journal of Econometrics* 50 (1-2): 15–29.

— (1993) 'Markets and Freedoms: Achievements and Limitations of the Market Mechanism in Promoting Individual Freedoms', *Oxford Economic Papers* 45 (4): 519–541.

Sugden, Robert (1998) 'The Metric of Opportunity', *Economics and Philosophy* 14 (2): 307–337.

— (2003) 'Opportunity as a Space for Individuality: Its Value and the Impossibility of Measuring It', *Ethics* 113 (4): 783–809.

van Hees, Martin (1998) 'On the Analysis of Negative Freedom', *Theory and Decision* 45 (2): 175–197.

— (2004) 'Freedom of Choice and Diversity of Options: Some Difficulties', *Social Choice and Welfare* 22 (1): 253–266.