The Axioms of Expected-Utility Theory

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Notation

We adopt the following standard notation:

'$x \succ y$' denotes that x preferred to y.

'$x \sim y$' denotes there being indifference between x and y.

'$x \succsim y$' denotes that x is at least as preferred as y.

'$x \# y$' denotes there being a preference gap between x and y.

'$[x, p; y, 1-p]$' denotes a lottery with a p probability of x and a $1-p$ probability of y.

The Axioms of Expected-Utility Theory

Transitivity
If $x \succsim y$ and $y \succsim z$, then $x \succsim z$.

Completeness
$x \succsim y$ or $y \succsim x$.

Independence
If $x \succ y$ and $0 < p \leq 1$, then $[x, p; z, 1-p] \succ [y, p; z, 1-p]$.

Continuity
If $x \succ y$ and $y \succ z$,
then there are numbers $0 < p < 1$ and $0 < q < 1$
such that $[x, p; z, 1-p] \succ y$ and $y \succ [x, q; z, 1-q]$.

Are we rationally required to satisfy these axioms?

The Transitivity Axiom

Transitivity
If $x \succsim y$ and $y \succsim z$, then $x \succsim z$.

Or, equivalently, without technical notation:

Transitivity
If x is at least as preferred as y and y is at least as preferred as z, then x is at least as preferred as z.

Transitivity
If $x \succsim y$ and $y \succsim z$, then $x \succsim z$.

R. Duncan Luce (1956, p. 179) presents the following counter-example:

Let c_0 be a cup of coffee with no sugar.
Let c_1 be a cup of coffee with one grain of sugar.
Let c_2 be a cup of coffee with two grains of sugar.

Suppose one can taste the difference between c_0 and c_2, and that one prefers c_2 to c_0.
But suppose furthermore that one cannot taste the difference between c_0 and c_1 or between c_1 and c_2, and that one therefore is indifferent between c_0 and c_1 and between c_1 and c_2.

These preferences seem rational, but we have:

$c_0 \succsim c_1$ and $c_1 \succsim c_2$, but not $c_0 \succsim c_2$.

The Puzzle of the Self-Torturer

Warren Quinn (1990, p. 79)

> *Suppose a special electric device has 1001 settings: 0, 1, 2, 3, . . . , 1000 and works as follows: moving up a setting raises, by a tiny increment, the amount of electric current applied to the self-torturer's body. The increments in current are so small that the self-torturer cannot tell the difference between adjacent settings. He can, however, tell the difference between settings that are far apart. Since the self-torturer cannot feel any difference in comfort between adjacent settings but gets $10,000 at each advance, he prefers, for any two consecutive settings s and s+1, stopping at s+1 to stopping at s. But, since he does not want to live in excruciating pain, even for a great fortune, he also prefers stopping at a low setting, such as 0, over stopping at a high setting, such as 1000.*

The Money-Pump Argument for Acyclicity

But before we consider an argument for Transitivity, we will first consider an argument for the weaker axiom acyclicity.

Acyclicity
Not ($x_1 \succ x_2$, $x_2 \succ x_3$, ..., and $x_n \succ x_1$).

Example of cyclic preferences:

$$\begin{array}{ccccc} a & & \succ & & b \\ & \succ & & \succ & \\ & & c & & \end{array}$$

Example of acyclic, but not transitive, preferences:

$$\begin{array}{ccccc} a & & \sim & & b \\ & \succ & & \succ & \\ & & c & & \end{array}$$

Aim of a money-pump argument:

> Show that any agent who violates a certain alleged requirement will in some possible situation be forced to violate a plausible dominance principle.

Alleged rational requirement:

> Rational preferences are acyclic.

Some standard dominance principles:

> **Synchronic Dominance**
> It is rationally required that one does not choose an alternative to which another alternative is preferred.
>
> **Diachronic Dominance**
> It is rationally required that one does not make a sequence of choices to which an alternative sequence of choices is preferred.

$a =$ full professor with a salary of \$5,000.
$b =$ associate professor at \$5,500.
$c =$ assistant professor at \$6,000.

$$\begin{array}{ccccc} a & & \succ & & b \\ & \succ & & \succ & \\ & & c & & \end{array}$$

Donald Davidson, J. C. C. McKinsey, Patrick Suppes (1955, p. 146)

> *The department head, advised of Mr. S's preferences, says, 'I see you prefer b to c, so I will let you have the associate professorship—for a small consideration. The difference must be worth something to you.' Mr. S. agrees to slip the department head \$25. to get the preferred alternative. Now the department head says, 'Since you prefer a to b, I'm prepared—if you will pay me a little for my trouble—to let you have the full professorship.' Mr. S. hands over another \$25. and starts to walk away, well satisfied, we may suppose. 'Hold on,' says the department head, 'I just realized you'd rather have c than a. And I can arrange that—provided...'*

The Sequential Interpretation

$$\begin{array}{ccccc} a & & \succ & & b \\ & \rotatebox{-45}{$\succ$} & & \rotatebox{45}{$\succ$} & \\ & & c & & \end{array}$$

Start with c.

– Pay for a swap from c to b?

Must accept, since c dominated by b.

– Swap from b to a?

Must accept, since b dominated by a.

– Swap from a back to c?

Must accept, since a dominated by c.

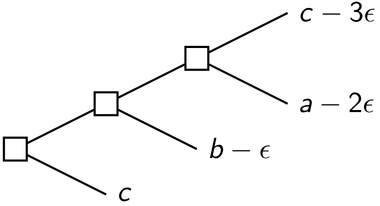
The agent now has the alternative he had at the start, but he is poorer than he would have been if he had turned down each swap.

Decision Trees

Sequential-choice situation are often represented by decision trees.

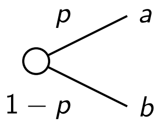
In these, choice nodes are represented by squares.

For example, the case from the last slide is represented as follows:



Chance nodes are represented by circles.

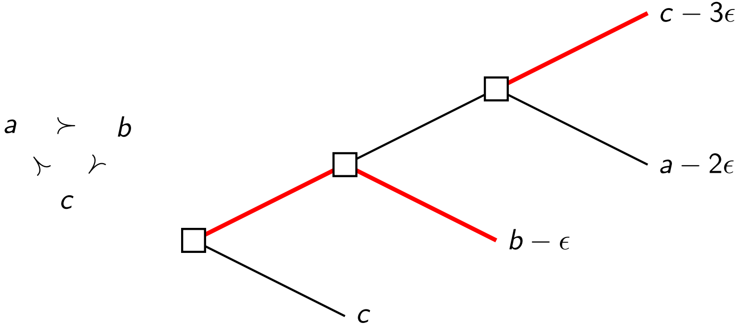
For example, the following represents the lottery $[a, p; b, 1-p]$:



Objection: Backwards Induction

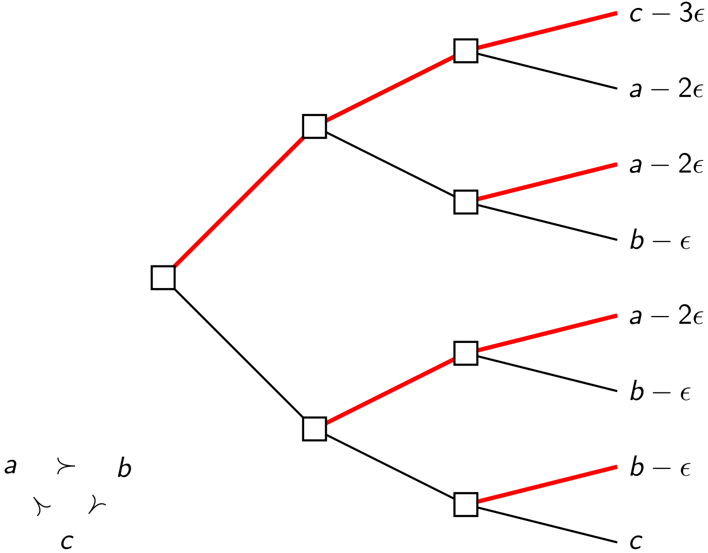
A standard objection is that one can, by using backwards induction, predict what future choice one would make and then take that into account when one chooses.

The thick red lines represents what choice one predicts that one would choose in a situation.



Taking what one choose in the future into account, one chooses $b - \epsilon$ and does not violate Diachronic Monetary Dominance. (Yet one still violates Diachronic Dominance).

Wlodek Rabinowicz (2000, p. 141) presents a money pump that works for agents that solve their decision problem by backwards induction.

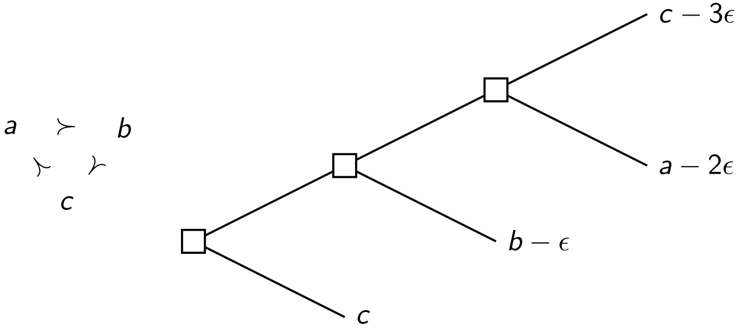


Objection: The agent can adopt a plan

Edward F. McClennen (1990, p. 13)

A resolute chooser is someone who

> *proceeds, against the background of his decision to adopt a particular plan, to do what the plan calls upon him to do, even though it is true (and he knows it to be true) that were he not committed to choosing in accordance with that plan, he would now be disposed to do something quite distinct from what the plan calls upon him to do.*



A problem with the resolute-choice objection is that, regardless of what plan one were to adopt, one would still violate the sequential-dominance principle, because all available sequences of choices are dominated.

A Synchronic Version of the Argument for Acyclicity

Cyclic preferences:
$$\begin{array}{ccccc} a & & \succ & & b \\ & \succ & & \succ & \\ & & c & & \end{array}$$

Offer a choice from the option set $\{a, b, c\}$.

Synchronic Dominance
A rational choice is one which selects an alternative to which none is preferred.

The Money-Pump Argument for Transitivity

Amos Tversky (1969, p. 45)

> *For if one violates transitivity, it is a well-known conclusion that he is acting, in effect, as a "money-pump." Suppose an individual prefers y to x, z to y, and x to z. It is reasonable to assume that he is willing to pay a sum of money to replace x by y. Similarly, he should be willing to pay some amount of money to replace y by z and still a third amount to replace z by x. Thus, he ends up with the alternative he started with but with less money.*

Tversky's method, however, assumes that there is a cycle of strict preferences. But there are other ways to violate Transitivity:

PPI-preferences:

$$\begin{array}{ccccc} a & & \succ & & b \\ & \rotatebox{-45}{$\sim$} & & \rotatebox{-135}{$\succ$} & \\ & & c & & \end{array}$$

PII-preferences:

$$\begin{array}{ccccc} a & & \succ & & b \\ & \rotatebox{-45}{$\sim$} & & \rotatebox{45}{$\sim$} & \\ & & c & & \end{array}$$

We ignore the possibility of preferential gaps for now, but we will deal with it later when we discuss Completeness.

PP#-preferences:

$$\begin{array}{ccccc} a & & \succ & & b \\ & \rotatebox{-45}{$\#$} & & \rotatebox{-135}{$\succ$} & \\ & & c & & \end{array}$$

PI#-preferences:

$$\begin{array}{ccccc} a & & \succ & & b \\ & \rotatebox{-45}{$\#$} & & \rotatebox{45}{$\sim$} & \\ & & c & & \end{array}$$

The Small-Bonus Approach

PII-preferences:

$$\begin{array}{ccccc} a & & \succ & & b \\ & \sim & & \sim & \\ & & c & & \end{array}$$

Make the agent prefer the swap from a to c by offering the agent a very small sum of money if he is willing to take c instead of a.

Let c^+ be c with an additional sum of money.

$$\begin{array}{ccccc} c^+ & & \succ & & c \\ & \overset{?}{\succ} & & \sim & \\ & & a & & \end{array}$$

A problem is that we cannot rely on Transitivity in an argument for Transitivity. And then it is unclear why the agent needs to prefer c^+ to a.

Dominance for Lotteries

If there is at least one positively probable state where the outcome of x is strictly preferred to the outcome of y and no state where the outcome of x is not weakly preferred to the outcome of y, then x is strictly preferred to y.

PPI-preferences:

$$\begin{tikzcd} a \arrow[rr, phantom, "\succ"] \arrow[dr, phantom, sloped, "\sim"] & & b \arrow[dl, phantom, sloped, "\succ"] \\ & c & \end{tikzcd}$$

PII-preferences:

$$\begin{tikzcd} a \arrow[rr, phantom, "\succ"] \arrow[dr, phantom, sloped, "\sim"] & & b \arrow[dl, phantom, sloped, "\sim"] \\ & c & \end{tikzcd}$$

	S_1	S_2	S_3
L_1	a	b	c
L_2	b	c	a
L_3	c	a	b

$$\begin{tikzcd} L_1 \arrow[rr, phantom, "\succ"] & & L_2 \arrow[dl, phantom, sloped, "\succ"] \\ & L_3 \arrow[ul, phantom, sloped, "\succ"] & \end{tikzcd}$$

Offer a choice from the option set $\{L_1, L_2, L_3\}$.

The Completeness Axiom

Completeness
$x \succsim y$ or $y \succsim x$.

Or, equivalently, without technical notation:

Completeness
Either x is at least as preferred as y or y is at least as preferred as x.

Ruth Chang (2002, p. 669)

> *Suppose you must determine which of a cup of coffee and a cup of tea tastes better to you. The coffee has a full-bodied, sharp, pungent taste, and the tea has a warm, soothing, fragrant taste. It is surely possible that you rationally judge that the cup of Sumatra Gold tastes neither better nor worse than the cup of Pearl Jasmine and that although a slightly more fragrant cup of the Jasmine would taste better than the original, the more fragrant Jasmine would not taste better than the cup of coffee.*

$$\begin{array}{ccc} \text{tea+} & & \\ \curlyvee & \not\prec & \\ \text{tea} & \not\succ \; \not\prec & \text{coffee} \end{array}$$

The Small-Improvement Argument

There exist things x, y, and z such that x is not preferred to y, y is not preferred to x, z is preferred to x, and z is not preferred to y.

For all things x, y, and z, if one is indifferent between x and y and z is preferred to x, then z is preferred to y.

So, there exist things x and y such that one is not indifferent between x and y, x is not preferred to y, and y is not preferred to x.

$$\begin{array}{ccc} z & & \\ \curlyvee & \not\prec & \\ x & \not\succ \; \not\prec & y \end{array}$$

Arguments for Completeness

A standard way of defining preference relations is in terms of a choice function as follows:

$$x \sim y =_{\mathrm{df}} C(\{x, y\}) = \{x, y\}$$
$$x \succ y =_{\mathrm{df}} C(\{x, y\}) = \{x\}$$
$$y \succ x =_{\mathrm{df}} C(\{x, y\}) = \{y\}$$

This suggest that preferential gaps are defined as:

$$x \# y =_{\mathrm{df}} C(\{x, y\}) = \emptyset$$

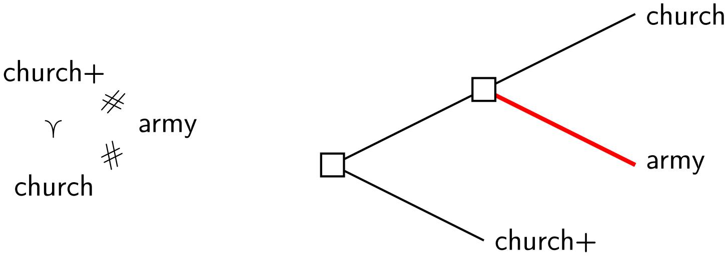
But is this possible on, for example, the following interpretations of the choice function?

$C(S) =_{\mathrm{df}}$ the set of options in S that are at least as desired as every option in S.

$C(S) =_{\mathrm{df}}$ the set of options in S that the agent is disposed to choose from S.

John Broome (1999, p. 156)

> *Suppose two careers are open to you: a career in the army and a good career as a priest. Suppose they are incommensurate in their goodness. Then choosing either would not be wrong. You have to choose without the guidance of reason, and suppose you choose the army: you commit yourself to the army career, and give up the chance of a good career in the church. In doing so you are doing nothing wrong. But then suppose another opportunity comes up to join the church, this time in much worse conditions. You now face a choice between the army or a much less good career as a priest. Suppose these two, also, are incommensurate. Choosing either would not be wrong You have to choose without the guidance of reason. Suppose this time you choose the church. Once again you do nothing wrong. But though you have not acted wrongly in either of your choices, the effect of the two together is that you end up with a much worse career in the church than you could have had.*

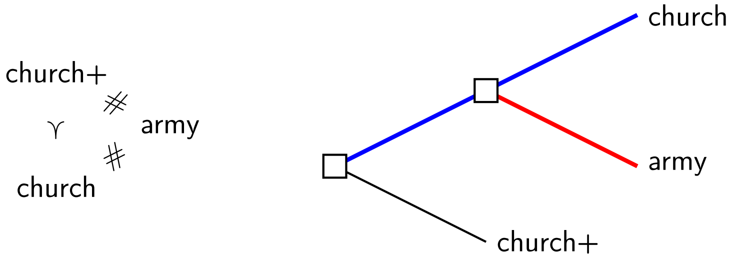


Yet one could avoid violating Diachronic Dominance if one refrained from choosing the church career in the second choice node.

One might perhaps argue that if it is rationally permitted to make the sequence of choices that ends with the army career, then it is rationally permitted to turn down the better church career.

But then when one faces the second choice it should be rationally permitted to choose the worse church career since it is not worse than the army career.

Hence it follows that it is rationally permitted to make the blue choices, which contradicts Diachronic Dominance.



Arguing for Transitivity without Assuming Completeness

$PP\#$-preferences:

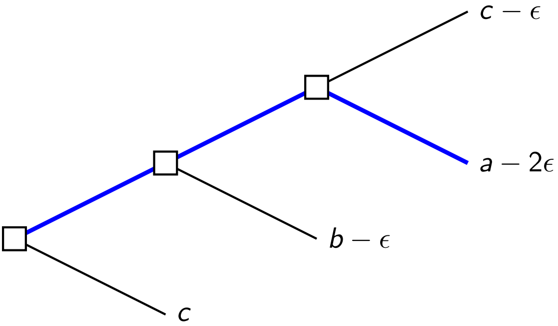
$$\begin{array}{ccccc} a & & \succ & & b \\ & \# & & \succ & \\ & & c & & \end{array}$$

$PI\#$-preferences:

$$\begin{array}{ccccc} a & & \succ & & b \\ & \# & & \sim & \\ & & c & & \end{array}$$

A Money-Pump Argument for Transitivity without Assuming Completeness?

$$\begin{array}{ccc} a & \succ & b \\ & \not\parallel \quad \succ & \\ & c & \end{array}$$



One can avoid violating Diachronic Dominance if one does not swap a for c, that is, if one makes the blue choices.

The Independence Axiom

Independence
If $x \succ y$ and $0 < p \leq 1$, then $[x, p; z, 1-p] \succ [y, p; z, 1-p]$.

Or, equivalently, without technical notation:

Independence
If x is preferred to y and p is a number greater than zero and lesser or equal to one, then a lottery with a probability p of x and a probability $1-p$ of z is preferred to a lottery with a probability p of y and a probability $1-p$ of z.

Allais's Paradox

Maurice Allais (1979, p. 89)

Situation 1. Choose between:

	Prize	Probability
Gamble 1	\$500,000	1
Gamble 2	\$2,500,000	0.1
	\$500,000	0.89
	\$0	0.01

Situation 2. Choose between:

	Prize	Probability
Gamble 3	\$500,000	0.11
	\$0	0.89
Gamble 4	\$2,500,000	0.1
	\$0	0.9

Many people prefer Gamble 1 to Gamble 2 and Gamble 4 to Gamble 3.

Independence

If $x \succ y$ and $0 < p \leq 1$, then $[x, p; z, 1-p] \succ [y, p; z, 1-p]$.

Prizes in units of \$100,000.
All tickets are equally likely to be drawn.

	Ticket Number	
	1	2–11
A	5	5
B	0	25

	Ticket Number		
	1	2–11	12–100
$[A, 0.12; C, 0.88]$	5	5	C
$[B, 0.12; C, 0.88]$	0	25	C

Independence requires that one prefers A to B if and only if one prefers $[A, 0.12; C, 0.88]$ to $[B, 0.12; C, 0.88]$, for any outcome C.

Leonard J. Savage (1954, p. 103)

Prizes in units of \$100,000.
All tickets are equally likely to be drawn.

		Ticket Number		
		1	2–11	12–100
Situation 1	Gamble 1	5	5	5
	Gamble 2	0	25	5
Situation 2	Gamble 3	5	5	0
	Gamble 4	0	25	0

Since the prizes are the same in each situation for both gambles for tickets 12–100, these tickets should not matter for the choice between the gambles.

And then we should prefer Gamble 1 to Gamble 2 if and only if we prefer Gamble 3 to Gamble 4.

Ellsberg's Paradox

Daniel Ellsberg (1961, pp. 650–651)

Suppose an urn contains 90 balls. 30 of them are red and the remaining 60 balls are either black or yellow. But the proportion of black and yellow balls is unknown.

Situation 1. Choose between:

Gamble 1	Receive \$100 if a red ball is drawn.
Gamble 2	Receive \$100 if a black ball is drawn.

Situation 2. Choose between:

Gamble 3	Receive \$100 if a red or yellow ball is drawn.
Gamble 4	Receive \$100 if a black or yellow ball is drawn.

Most people prefer Gamble 1 to Gamble 2 and Gamble 4 to Gamble 3, since Gamble 1 and Gamble 4 are the only ones with known probabilities.

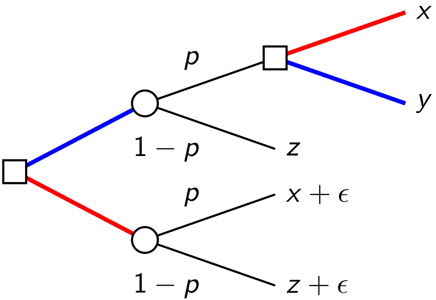
		30	60	
		Red	Black	Yellow
Situation 1	Gamble 1	\$100	\$0	\$0
	Gamble 2	\$0	\$100	\$0
Situation 2	Gamble 3	\$100	\$0	\$100
	Gamble 4	\$0	\$100	\$100

Preferring Gamble 1 to Gamble 2 and Gamble 4 to Gamble 3 violates Independence.

A Money-Pump Argument for Independence

Suppose one violates Independence by preferring
$x \succ y$ and $[y, p; z, 1-p] \succ [x, p; z, 1-p]$.

Given that ϵ is a sufficiently small bonus, its plausible that
$[y, p; z, 1-p] \succ [x+\epsilon, p; z+\epsilon, 1-p]$.



By backwards induction, one chooses (red lines)
$[x+\epsilon, p; z+\epsilon, 1-p]$.

But one could have chosen (blue lines) $[y, p; z, 1-p]$.

The Continuity Axiom

Continuity
If $x \succ y$ and $y \succ z$,
then there are numbers $0 < p < 1$ and $0 < q < 1$
such that $[x, p; z, 1-p] \succ y$ and $y \succ [x, q; z, 1-q]$.

Or, equivalently, without technical notation:

Continuity
If x is preferred to y and y is preferred to z, then there is a p and q greater than zero and less than one such that a lottery with a probability p of x and a probability $1 - p$ of z is preferred to y and y is preferred to a lottery with a probability q of x and a probability $1 - q$ of z.

The Alchain Objection

Continuity
If $x \succ y$ and $y \succ z$,
then there are numbers $0 < p < 1$ and $0 < q < 1$
such that $[x, p; z, 1-p] \succ y$ and $y \succ [x, q; z, 1-q]$.

Armen A. Alchain (1953, pp. 36–37) presents the following counter-example:

> *Suppose C is two bars of candy, B is one bar of candy, and A is being shot in the head. Form an uncertain prospect of C and A with probability p for C. If there is no p, however small or close to zero, which could possibly make one indifferent between the uncertain prospect and B, the one bar of candy, he is rejecting [the continuity axiom].*

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